ON THE STABILITY OF SOLUTIONS OF THE LINEARIZED PLASMA EQUATION
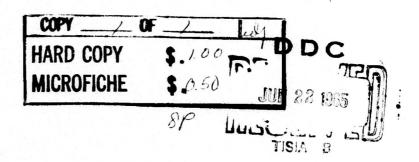
Richard Bellman
Mathematics Division
The RAND Corporation

John M. Richardson
Research Laboratories
Hughes Aircraft Company

P–1898

2 February 1960

Approved for OTS release

SUMMARY

We consider the equation

$$u_t + ixu - iaf'(x) \int_{-\infty}^{\infty} u\,dx = 0, \quad u(x,0) = h(x),$$

a linearized version of a basic equation in plasma physics.
Deriving a relation equivalent to the usual dispersion
relation, but of more convenient form, we derive some simple
criteria for stability of solutions of this equation.

ON THE STABILITY OF SOLUTIONS OF THE
LINEARIZED PLASMA EQUATION

Richard Bellman
John M. Richardson

1. Introduction

The equation

$$(1) \qquad u_t + ixu - iaf'(x)\int_{-\infty}^{\infty} u\,dx, \quad u(x,0) = h(x),$$

is a linearized version of a basic equation in plasma physics. Here $u = u(x,t)$, $f(x)$ is a real function of x and a is a positive constant. A problem of some importance is that of determining when the solutions of (1) are uniformly bounded as $t \rightarrow \infty$.

The usual approach to this problem in the physical literature is to write

$$(2) \qquad u = e^{st}\phi(x),$$

where we normalize by the condition $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Then (1) yields

$$(3) \qquad \phi(x) = \frac{iaf'(x)}{s + ix}.$$

The normalization condition yields the "characteristic equation" for s,

$$(4) \qquad ia\int_{-\infty}^{\infty} \frac{f'(x)dx}{s + ix} = 1,$$

called the "dispersion relation." Since it is generally rather

difficult to study the distribution of the roots of an equation of this type, we wish to present a different approach which will enable us to present some simple conditions for stability.

2. Reduction to Renewal Equation

Let us write (1.1) in the form

(1) $\quad u_t + ixu = iaf'(x) \int_{-\infty}^{\infty} u\,dx, \quad u(x,0) = h(x),$

and regard the right-hand side as a forcing term. Then we may write

(2) $\quad u = e^{-ixt}h(x) + iaf'(x) \int_0^t e^{-ix(t-s)} \left[\int_{-\infty}^{\infty} u(x_1,t)\,dx_1 \right] ds.$

Integrating over $[-\infty,\infty]$, we have

(3) $\quad \int_{-\infty}^{\infty} u\,dx = \int_{-\infty}^{\infty} e^{-ixt}h(x)\,dx$

$$+ ia \int_0^t \left[\int_{-\infty}^{\infty} f'(x)e^{-ix(t-s)}\,dx \right] \left[\int_{-\omega}^{\infty} u\,dx_1 \right] ds.$$

Let us now introduce some new functions:

(4) $\quad v(t) = \int_{-\infty}^{\infty} u(x,t)\,dx,$

$$H(t) = \int_{-\infty}^{\infty} e^{-ixt}h(x)\,dx,$$

$$K(t) = \int_{-\infty}^{\infty} f'(x)e^{-ixt}\,dx.$$

Then

(5) $$v(t) = H(t) + \int_0^t K(t - s)v(s)ds,$$

an integral equation of renewal type.

The Laplace transform readily yields the equation

(6) $$L(v) = \frac{L(H)}{1 - L(K)},$$

or, using the Laplace inverse,

(7) $$v = \frac{1}{2\pi i} \int_C \frac{L(H)e^{st}ds}{1 - L(K)},$$

where C can be taken to be a line parallel to the imaginary axis and sufficiently far to the right.

The question of the asymptotic behavior of $u(x,t)$ as $t \rightarrow \infty$ thus reduces to the problem of determining the root with largest real part of the equation

(8) $$1 = \int_0^\infty e^{-st}K(t)dt.$$

3. Discussion

Using the definition of $K(t)$ given in (2.4), we see that (2.5) can be written

(1) $$1 = \int_0^\infty e^{-st}\left[ia\int_{-\infty}^\infty f'(\lambda)e^{-i\lambda t}dx\right]dt$$

$$= ia\int_{-\infty}^\infty \frac{f'(x)dx}{s + ix},$$

if we interchange the orders of integration. However, we
definitely do not wish to do this. The advantage of using
(2.5) rather than the dispersion relation lies in the fact
that much more is known about the roots of equations of the
form of (2.5) than the roots of equations such as (1.4). We
shall illustrate this in the next section.

4. Some Simple Criteria

For the purpose of stability investigations, we wish to
obtain simple conditions which ensure that (2.5) possesses no
roots with nonnegative real part. Although a simple criterion
is

(1) $$\int_0^\infty |K(t)|\,dt < 1,$$

unless $K(t) \geq 0$ this is much too crude for useful application.

Let us suppose that $K(t)$ is real, and write $s = \sigma + i\tau$.
Then (2.5) yields two equations,

(2) $$1 = \int_0^\infty e^{-\sigma t} \cos t\tau K(t)\,dt,$$

$$0 = \int_0^\infty e^{-\sigma t} \sin t\tau K(t)\,dt.$$

Given a particular function $K(t)$ of reasonable analytic
structure, these equations can be used to determine stability
or nonstability.

A particular useful and simple criterion is the following

Lemma. If $g(t) > 0$ and monotone decreasing, then

(3) $\qquad \int_0^\infty g(t)\sin t\tau\,dt > 0$

<u>if</u> $\tau > 0.$

Write

(4) $\qquad \int_0^\infty g(t)\sin t\tau\,dt = \int_0^{\pi/\tau} \sin t\tau \Big[g(t) - g(t + \tfrac{\pi}{\tau})$

$$+ g(t + \tfrac{2\pi}{\tau}) - \cdots \Big] dt.$$

We see that the result actually holds under the weaker condition

(5) $\qquad g(t) - g(t + \tfrac{\pi}{\tau}) + g(t + \tfrac{2\pi}{\tau}) - g(t + \tfrac{3\pi}{\tau}) + \cdots \geq 0$

for $0 \leq t \leq \pi/\tau.$

5. An Application

As an application of the foregoing result, let us consider the important case where $f(x) = e^{-x^2}$. Since

(1) $\qquad \int_{-\infty}^\infty e^{-x^2 - ixt}dx = \sqrt{\pi}\, e^{-t^2/4},$

we have

(2) $\qquad \int_{-\infty}^\infty (-2ixe^{-x^2})e^{-ixt}dx = 2\sqrt{\pi}\, \tfrac{\partial}{\partial t}(e^{-t^2/4})$

$$= -\sqrt{\pi}\, t e^{-t^2/4}.$$

Hence the equation in (2.5) takes the form

(3) $\qquad 1 = -a\sqrt{\pi} \int_0^\infty t e^{-st} e^{-t^2/4}dt.$

Integration by parts yields

(4) $\int_0^\infty e^{-st}te^{-t^2/4}dt = 2 - 2s\int_0^\infty e^{-st}e^{-t^2/4}dt.$

Thus the equation in (3) becomes

(5) $\dfrac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}s} = \int_0^\infty e^{-st}e^{-t^2/4}dt.$

Write $s = \sigma + i\tau$ and equate real and complex parts. The resulting two equations are

(6) $\left(\dfrac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right)\dfrac{\sigma}{\sigma^2 + \tau^2} = \int_0^\infty e^{-\sigma t}e^{-t^2/4}\cos t\tau dt,$

$-\left(\dfrac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right)\dfrac{\tau}{\sigma^2 + \tau^2} = \int_0^\infty e^{-\sigma t}e^{-t^2/4}\sin t\tau dt.$

Without loss of generality take $\tau \geq 0$. Consider first the case $\tau > 0$. Then if $\sigma \geq 0$, the function $e^{-\sigma t}e^{-t^2/4}$ is positive and monotone decreasing for $t \geq 0$. Hence the second equation in (6) cannot hold for a nonnegative value of σ.

If $\tau = 0$, the first equation yields

(7) $\left(\dfrac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right)\dfrac{1}{\sigma} = \int_0^\infty e^{-\sigma t}e^{-t^2/4}dt,$

a relation which holds for no value of σ since

(8) $\dfrac{1}{\sigma} < \left(\dfrac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right)\dfrac{1}{\sigma} = \int_0^\infty e^{-\sigma t}e^{-t^2/4}dt < \int_0^\infty e^{-\sigma t}dt = \dfrac{1}{\sigma}.$

In the case of general $f(x)$, we obtain criteria in terms of the Fourier transform of $f'(x)$, or of $f(x)$. In this way we can exhibit families of functions which yield stability.